

Projective dimension and regularity of the path ideal of the line graph [†]

Guangjun Zhu

Abstract: By generalizing the notion of the path ideal of a graph, we study some algebraic properties of some path ideals associated to a line graph. We show that the quotient ring of these ideals are always sequentially Cohen-Macaulay and also provide some exact formulas for the projective dimension and the regularity of these ideals. As some consequences, we give some exact formulas for the depth of these ideals.

Keywords: projective dimension, castelnuovo-Mumford regularity, path ideal, sequentially Cohen-Macaulay, line graph.

Mathematics Subject Classifications (2010): 13D02; 13F55; 13C15; 13D99.

1. INTRODUCTION

The path ideal of a graph was first introduced by Conca and De Negri [4]. Fix an integer $m \geq 2$, and suppose that Γ is a directed graph with vertex set $V = \{x_1, \dots, x_n\}$, i.e., each edge has been assigned a direction. A sequence of m vertices x_{i_1}, \dots, x_{i_m} is said to be a path of length m if there are $m - 1$ distinct edges e_1, \dots, e_{m-1} such that $e_j = (x_{i_j}, x_{i_{j+1}})$ is a directed edge from x_{i_j} to $x_{i_{j+1}}$. By identifying the vertices with the variables in the polynomial ring $R = k[x_1, \dots, x_n]$ over a field k , the path ideal of Γ of length m is the monomial ideal

$$J_m(\Gamma) = (\{x_{i_1} \cdots x_{i_m} \mid x_{i_1}, \dots, x_{i_m} \text{ is a path of length } m \text{ in } \Gamma\})$$

Note that when $m = 2$, then $J_2(\Gamma)$ is simply the edge ideal of Γ , which is defined by Villarreal in [17]. Other higher dimensional analogues can be found in [7, 11], among others. The underlying theme in all correspondences is to relate the algebraic properties to the combinatorial properties, and vice versa. We mainly study the algebraic properties of the path ideal.

Path ideals appeared in [4] as an example of a family of monomial ideals that are generated by M -sequences. Among other things, it is shown that when Γ is a directed tree, the Rees algebra $\mathcal{R}(J_m(\Gamma))$ is normal and Cohen-Macaulay. The path ideals of complete bipartite graphs are shown to be normal in [14], while the path ideals of cycles are shown to have linear type in [3]. In [12], He and Tuyl study $J_m(\Gamma)$ in the special case that Γ is the line graph L_n . The line graph L_n is a graph with vertex set $V = \{x_1, \dots, x_n\}$ and directed edges $e_j = (x_j, x_{j+1})$ for $j = 1, \dots, n - 1$. Thus, the graph L_n has the form



[†]Supported by the National Natural Science Foundation of China (11271275) and by Foundation of Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-aged Teachers and Presidents and by Foundation of the Priority Academic Program Development of Jiangsu Higher Education Institutions.

School of Mathematical Sciences, Soochow University, Suzhou 215006, P. R. China, e-mail: zhuguangjun@suda.edu.cn

They prove that $R/J_m(L_n)$ is sequentially Cohen-Macaulay and also provide an exact formula for the projective dimension of $J_m(L_n)$ in terms of m and n . They showed that:

Theorem 1.1. (Theorem 4.1) *Let p, m, n, d be integers such that $n = p(m+1) + d$, where $p \geq 0$, $0 \leq d \leq m$ and $2 \leq m \leq n$. Then the projective dimension of $J_m(L_n)$ is given by*

$$pd(J_m(L_n)) = \begin{cases} 2p-1 & d \neq m; \\ 2p & d = m. \end{cases}$$

In [1], using purely combinatorial arguments, Alilooee and Faridi also gave the above formula for projective dimension of $J_m(L_n)$. Furthermore, they gave an explicit formula for Castelnuovo-Mumford regularity of $J_m(L_n)$ in terms of m and n . They showed that:

Theorem 1.2. (Corollary 4.14) *Let p, m, n, d be integers such that $n = p(m+1) + d$, where $p \geq 0$, $0 \leq d \leq m$ and $2 \leq m \leq n$. Then the regularity of $J_m(L_n)$ is given by*

$$reg(J_m(L_n)) = \begin{cases} p(m-1) + 1 & d \neq m; \\ p(m-1) + m & d = m. \end{cases}$$

We generalize the notion of the path ideal as the following: Let Γ be a directed graph with vertex set $V = \{x_1, \dots, x_n\}$, the path ideal of Γ of length m is the monomial ideal

$$I_{m,k}(\Gamma) = (u_1, \dots, u_k), \text{ where } u_1, \dots, u_k \text{ are some paths of length } m \text{ in } \Gamma.$$

When u_1, \dots, u_k are all paths of length m in Γ , $I_{m,k}(\Gamma) = J_m(\Gamma)$.

To the best of our knowledge, little is known about these ideals. It is, therefore, of interest to determine algebraic properties of the ideals $I_{m,k}(\Gamma)$. In this article we shall focus on the case that Γ is the line graph L_n and $I_{m,k}(L_n) = (u_1, \dots, u_k)$, where for any $1 \leq i \leq k$, $u_i = \prod_{j=1}^m x_{(i-1)(m-l)+j}$ is a path of length m in L_n and $1 \leq l \leq m$ is an integer. we shall abuse notation and write $I_{m,k}(L_n)$ for $I_{m,l,k}$. In Section 2, we study algebraic properties of the ideal $I_{m,l,k}$ and show that $R/I_{m,l,k}$ is sequentially Cohen-Macaulay. In Section 3, using the notion of a Betti-splitting, as defined in [9], we derive some exact formulas for the projective dimension and regularity of the ideal $I_{m,l,k}$ (see Theorems 3.5, 3.7 and 3.10). As some consequences, we give some exact formulas for the depth of these ideals.

2. PRELIMINARIES

In this section, we will show that the ideal $I_{m,l,k}$ can be viewed as the facet ideal of the simplicial complex $\Delta_{m,l,k}$ or the edge ideal of the clutter $\mathcal{C}_{m,l,k}$. By proving $\mathcal{C}_{m,l,k}$ has the free vertex property, we can obtain that the quotient ring $R/I_{m,l,k}$ is sequentially Cohen-Macaulay. We recall the relevant definitions.

Definition 2.1. *A simplicial complex Δ on the vertex set V is a collection of subsets of V with the property that if $F \in \Delta$ then all subsets of F are also in Δ . An element of Δ is called a face, the dimension of a face F is $|F| - 1$, and the dimension of Δ is the largest dimension of faces of Δ . The maximal faces of Δ under inclusion are called facets, and the set of facets of Δ is denoted by $\text{Facets}(\Delta)$. Simplicial complex Δ is called pure if all of its facets have the same dimension, otherwise Δ is non-pure. If $\text{Facets}(\Delta) = \{F_1, \dots, F_q\}$ we write $\Delta = \langle F_1, \dots, F_q \rangle$.*

Definition 2.2. *A clutter \mathcal{C} on vertex set V is a family of subsets of V , called edges, none of which is included in another. The set of vertices and edges of \mathcal{C} are denoted by $V_{\mathcal{C}}$ and $E_{\mathcal{C}}$ respectively.*

Given a clutter $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$, we can associate to \mathcal{C} the simplicial complex $\Delta = \{F \subseteq V_{\mathcal{C}} \mid F \subseteq E_i \text{ for some } E_i \in E_{\mathcal{C}}\}$. Conversely, given a simplicial complex Δ with vertex set V and set of facets $\text{Facets}(\Delta)$, we can associate to Δ the clutter $\mathcal{C} = (V, \text{Facets}(\Delta))$.

Let I be a squarefree monomial ideal of R with minimal generators x^{v_1}, \dots, x^{v_q} . We use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \dots, a_n) \in \mathbf{N}^n$. Note that the entries of each v_i are in $\{0, 1\}$. We associate to the ideal I a clutter \mathcal{C} by taking the set of indeterminates $V_{\mathcal{C}} = \{x_1, \dots, x_n\}$ as the vertex set and $E_{\mathcal{C}} = \{S_1, \dots, S_q\}$ as the edge set, where $S_i = \text{supp}(x^{v_i})$ is the support of x^{v_i} , i.e., S_i is the set of variables that occur in x^{v_i} . For this reason I is called the edge ideal of \mathcal{C} and is denoted $I = I(\mathcal{C})$. Edge ideal of a clutter is also called facet ideal because $\{S_1, \dots, S_q\}$ is exactly the set of facets of the simplicial complex $\Delta = \langle S_1, \dots, S_q \rangle$ generated by S_1, \dots, S_q .

Let Δ be a simplicial complex and $\sigma \in \Delta$, the deletion of σ from Δ is the simplicial complex defined by $\Delta \setminus \sigma = \{\tau \in \Delta \mid \sigma \not\subseteq \tau\}$, when $\sigma = \{x\}$, we shall abuse notation and write $\Delta \setminus x$ for $\Delta \setminus \{x\}$. If $\Delta = \langle F_1, \dots, F_q \rangle$, the simplicial complex obtained by removing the facet F_i from Δ is the simplicial complex $\Delta \setminus \langle F_i \rangle = \langle F_1, \dots, \hat{F}_i, \dots, F_q \rangle$.

The following definition of shellable is due to Björner and Wachs [2] and is usually referred to as nonpure shellable, here we drop the adjective “nonpure”.

Definition 2.3. *A simplicial complex Δ is shellable if the facets of Δ can be ordered F_1, \dots, F_s such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $l \in \{1, \dots, j-1\}$ with $F_j \setminus F_l = \{x\}$. We call F_1, \dots, F_s a shelling of Δ when the facets have been ordered with respect to the shellable definition.*

If the simplicial complex Δ is pure and satisfies the above definition of shellable, we will say Δ is pure shellable.

Definition 2.4. *Let M be a graded R -module. M is called sequentially Cohen-Macaulay if there exists a filtration of graded R -submodules of M*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each quotient M_i/M_{i-1} is Cohen-Macaulay and the Krull dimensions of the quotients are increasing, i.e., $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$.

A simplicial complex is said to be sequentially Cohen-Macaulay if its Stanley-Reisner ideal has a sequentially Cohen-Macaulay quotient.

It is well known that shellable implies sequentially Cohen-Macaulay.

Let \mathcal{C} be a clutter with vertex set V . A vertex cover of \mathcal{C} is a subset A of V that intersects every edge of \mathcal{C} . If A is a minimal element (under inclusion) of the set of vertex covers of \mathcal{C} , it is called a minimal vertex cover. To a squarefree monomial ideal $I = I(\mathcal{C})$, it also corresponds to a simplicial complex via the Stanley-Reisner correspondence [16]. We let $\Delta_{\mathcal{C}}$ represent this simplicial complex. Note that F is a facet of $\Delta_{\mathcal{C}}$ if and only if $V \setminus F$ is a minimal vertex cover of \mathcal{C} . As for clutters, we may say that the clutter \mathcal{C} is shellable if $\Delta_{\mathcal{C}}$ is shellable.

Definition 2.5. *Let $I' \subsetneq I$ be two ideals of R , I' is called a minor of I if there is a subset $V' = \{x_{i_1}, \dots, x_{i_r}, x_{j_1}, \dots, x_{j_s}\}$ of the set of variables $V = \{x_1, \dots, x_n\}$ such that I' is a proper ideal of $R' = k[V \setminus V']$ that can be obtained from the generator set of I by setting $x_{i_k} = 0$ and $x_{j_l} = 1$ for all k, l . A minor of \mathcal{C} is a clutter \mathcal{C}' on the vertex set*

$V_{C'} = V \setminus V'$ that corresponds to a minor $(0) \subsetneq I' \subsetneq R'$. The edges of C' are obtained from I' by considering the unique set of squarefree monomials of R' that minimally generate I' . For use below we say x_i is a free variable (resp. free vertex) of I (resp. C) if x_i only appears in one of the monomials x^{v_1}, \dots, x^{v_a} (resp. in one of the edges of C). If all the minors of C have free vertices, we say that C has the free vertex property. Note that if C has the free vertex property, then so do all of its minors.

Tuyl and Villarreal in [15] (also see in [18]) showed that the clutter with the free vertex property is shellable.

Theorem 2.6. *If a clutter C with the free vertex property, then Δ_C is shellable.*

The squarefree monomial ideal $I_{m,l,k}$ corresponds to a clutter (resp. simplicial complex), say $\mathcal{C}_{m,l,k}$ (resp. $\Delta_{m,l,k}$), its edges are precisely some such paths of length m in the line graph L_n . That is, $Ec_{m,l,k} = \{\{x_1, \dots, x_m\}, \{x_{(m-l)+1}, \dots, x_{2(m-l)+l}\}, \dots, \{x_{(k-1)(m-l)+1}, \dots, x_{k(m-l)+l}\}\}$. Throughout this paper, we will assume that the clutter with edge set $Ec_{m,l,k} = \{\{x_1, \dots, x_m\}, \{x_{(m-l)+1}, \dots, x_{2(m-l)+l}\}, \dots, \{x_{(k-1)(m-l)+1}, \dots, x_{k(m-l)+l}\}\}$ where l is an integer such that $1 \leq l \leq m$. This set corresponds to a squarefree monomial ideal $I_{m,l,k}$, which is the path ideal of the line graph L_n , i.e., $I_{m,l,k} = I(\mathcal{C}_{m,l,k})$.

Combining Definition 2.5 and Theorem 2.6, we then get the following proposition.

Proposition 2.7. *Let k, l, m be positive integers, $\mathcal{C}_{m,l,k}$ be a clutter with edge set $Ec_{m,l,k} = \{E_1, \dots, E_k\}$ where $E_i = \{x_{(i-1)(m-l)+1}, x_{(i-1)(m-l)+2}, \dots, x_{(i-1)(m-l)+m}\}$ for $i = 1, \dots, k$ and $I_{m,l,k} = I(Ec_{m,l,k})$ be the edge ideal of the clutter $\mathcal{C}_{m,l,k}$. Then the quotient ring $R/I_{m,l,k}$ is sequentially Cohen-Macaulay.*

Proof. By theorem 2.6, it is enough to prove that $\mathcal{C}_{m,l,k}$ has the free vertex property. Let $V = \{x_{i_1}, \dots, x_{i_r}, x_{j_1}, \dots, x_{j_s}\}$ be any subset of the set of variables $\{x_1, \dots, x_{k(m-l)+l}\}$ and $R' = k[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_r}}, \dots, \widehat{x_{j_1}}, \dots, \widehat{x_{j_s}}, \dots, x_{k(m-l)+l}]$. One can assume that I' is an ideal of R' minimally generated by monomials $u'_{l_1}, \dots, u'_{l_t}$ with $l_1 < l_2 < \dots < l_t$, and $x_{i_a} \nmid u'_{l_b}$ for any $1 \leq a \leq r$, $1 \leq b \leq t$, and for any $1 \leq b \leq t$, u'_{l_b} is obtained by dividing $u_{l_b} = \prod_{j=1}^m x_{(l_b-1)(m-l)+j}$ by the product of all the x_{j_c} such that $c \in \{1, \dots, s\}$. Set $a_j = (l_1 - 1)(m - l) + j$ for $j = 1, \dots, m$ and $d = \min\{a_j \mid j \in \{1, \dots, m\} \text{ and } a_j \notin \{j_1, \dots, j_s\}\}$. It is obvious that x_d is a free variable of I' and the proof is completed. \square

3. PROJECTIVE DIMENSION AND REGULARITY OF THE IDEAL $I_{m,l,k}$

In this section, we will provide some formulas for computing the projective dimension and the regularity of $I_{m,l,k}$. As some consequences, we also give some exact formulas for the depth of $I_{m,l,k}$.

Let M be a graded R -module where $R = K[x_1, \dots, x_n]$. Associated to M is a minimal graded free resolution of the form

$$0 \rightarrow \bigoplus_j R(-j)^{\beta_{p,j}(M)} \rightarrow \bigoplus_j R(-j)^{\beta_{p-1,j}(M)} \rightarrow \dots \rightarrow \bigoplus_j R(-j)^{\beta_{0,j}(M)} \rightarrow M \rightarrow 0, \text{ where}$$

the maps are exact, $p \leq n$, and $R(-j)$ is the R -module obtained by shifting the degrees of R by j . The number $\beta_{i,j}(M)$, the (i, j) -th graded Betti number of M , is an invariant of M that equals the number of minimal generators of degree j in the i th syzygy module of M .

Of particular interest are the following invariants which measure the size of the minimal graded free resolution of I . The projective dimension of I , denoted $\text{pd}(I)$, is defined to be

$$\text{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

The regularity of I , denoted $\text{reg}(I)$, is defined by

$$\text{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

We now derive some formulas for $\text{pd}(I_{m,l,k})$ and $\text{reg}(I_{m,l,k})$ in some special cases by using some tools developed in [9]. We let $\mathcal{G}(I)$ denote the unique set of minimal generators of a monomial ideal I .

Definition 3.1. *Let I be a monomial ideal, and suppose that there exists monomial ideals J and K such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I = J + K$ is a Betti splitting if*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0,$$

where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.

This formula was first obtained for the total Betti numbers by Eliahou and Kervaire [5] and extended to the graded case by Fatabbi [8]. In the article [9], the authors describe a number of sufficient conditions for an ideal I to have a Betti splitting. We shall require the following such condition.

Theorem 3.2. ([9, Corollary 2.7]). *Suppose that $I = J + K$ where $\mathcal{G}(J)$ contains all the generators of I divisible by the variable x_i and $\mathcal{G}(K)$ is a nonempty set containing the remaining generators of I . If J has a linear resolution, then $I = J + K$ is a Betti splitting.*

When $I = J + K$ is a Betti splitting ideal, Definition 3.1 implies the following result:

Corollary 3.3. *If $I = J + K$ is a Betti splitting, then*

- (i) $\text{reg}(I) = \max\{\text{reg}(J), \text{reg}(K), \text{reg}(J \cap K) - 1\}$,
- (ii) $\text{pd}(I) = \max\{\text{pd}(J), \text{pd}(K), \text{pd}(J \cap K) + 1\}$.

We need the following Lemma:

Lemma 3.4. *Let $R_1 = k[x_1, \dots, x_m]$ and $R_2 = k[x_{m+1}, \dots, x_n]$ be two polynomial rings, $I \subseteq R_1$ and $J \subseteq R_2$ be two nonzero homogeneous ideals. Then*

- (1) $\text{pd}(I + J) = \text{pd}(I) + \text{pd}(J) + 1$,
- (2) $\text{reg}(I + J) = \text{reg}(I) + \text{reg}(J) - 1$,
- (3) $\text{reg}(IJ) = \text{reg}(I) + \text{reg}(J)$.

Proof. Let $R = k[x_1, \dots, x_n]$. Then, by Proposition 2.2.20 of [16], we have that $R/I + J \cong R_1/I \otimes_k R_2/J$. Hence we get that $\text{pd}(R/I + J) = \text{pd}(R_1/I) + \text{pd}(R_2/J)$. It follows that

$$\begin{aligned} \text{pd}(I + J) &= \text{pd}(R/I + J) - 1 = \text{pd}(R_1/I) + \text{pd}(R_2/J) - 1 \\ &= (\text{pd}(I) + 1) + (\text{pd}(J) + 1) - 1 = \text{pd}(I) + \text{pd}(J) + 1, \end{aligned}$$

As for the second and the third assertion, by Lemma 3.2 of [13], we obtain that $\text{reg}(R/I + J) = \text{reg}(R_1/I) + \text{reg}(R_2/J)$ and $\text{reg}(R/IJ) = \text{reg}(R_1/I) + \text{reg}(R_2/J) + 1$. Therefore, we can conclude that

$$\begin{aligned} \text{reg}(I + J) &= \text{reg}(R/I + J) + 1 = \text{reg}(R_1/I) + \text{reg}(R_2/J) + 1 \\ &= (\text{reg}(I) - 1) + (\text{reg}(J) - 1) + 1 = \text{reg}(I) + \text{reg}(J) - 1, \end{aligned}$$

and

$$\begin{aligned}\operatorname{reg}(IJ) &= \operatorname{reg}(R/IJ) + 1 = \operatorname{reg}(R_1/I) + \operatorname{reg}(R_2/J) + 2 \\ &= (\operatorname{reg}(I) - 1) + (\operatorname{reg}(J) - 1) + 2 = \operatorname{reg}(I) + \operatorname{reg}(J).\end{aligned}$$

We finished the proof. \square

Now, we prove some main results of this section.

Theorem 3.5. *Let k, l, m, n be integers such that $n = k(m - l) + l$ where $k \geq 1$, $m \geq 2$ and $l < \lceil \frac{m}{2} \rceil$, here $\lceil \frac{m}{2} \rceil$ denotes the smallest integer $\geq \frac{m}{2}$. Let $I_{m,l,k} = (u_1, \dots, u_k)$ with $u_i = \prod_{j=1}^m x_{(i-1)(m-l)+j}$ for any $1 \leq i \leq k$. Then $\operatorname{pd}(I_{m,l,k}) = k - 1$, $\operatorname{reg}(I_{m,l,k}) = (k - 1)(m - l - 1) + m$.*

Proof. We first claim that $m - 2l - 1 \geq 0$. In fact, if $m = 2s + 1$, then $\lceil \frac{m}{2} \rceil = s + 1$. By the hypothesis, we have that $2l + 1 \leq 2(\lceil \frac{m}{2} \rceil - 1) + 1 = 2s + 1 = m$. On the other hand, if $m = 2s$, then $\lceil \frac{m}{2} \rceil = s$. Thus $2l + 1 \leq 2(\lceil \frac{m}{2} \rceil - 1) + 1 = 2s - 1 < m$. This proves the claim. We prove these assertions by induction on k . It is clear for $k = 1$. If $k = 2$, we let $J_1 = I_{m,l,1}$ and $K_1 = (u_2)$, which contains all the generators of $I_{m,l,2}$ divisible by the variable x_{2m-l} . Because K_1 has a linear resolution, $I_{m,l,2} = J_1 + K_1$ is a Betti splitting by Theorem 3.2 and $J_1 \cap K_1 = K_1(\prod_{j=1}^{m-l} x_j)$. Note that J_1, K_1 and $J_1 \cap K_1$ are principal ideals, which implies that $\operatorname{pd}(J_1) = \operatorname{pd}(K_1) = \operatorname{pd}(J_1 \cap K_1) = 0$. Thus, by Corollary 3.3, we obtain that

$$\operatorname{pd}(I_{m,l,2}) = \max\{\operatorname{pd}(J_1), \operatorname{pd}(K_1), \operatorname{pd}(J_1 \cap K_1) + 1\} = 1.$$

Because the variables that appear in K_1 and $(\prod_{j=1}^{m-l} x_j)$ are different, $\operatorname{reg}(J_1 \cap K_1) = \operatorname{reg}(J_1) + \operatorname{reg}(K_1) = m + (m - l)$ by Lemma 3.4. Therefore, by Corollary 3.3, we can conclude that

$$\begin{aligned}\operatorname{reg}(I_{m,l,2}) &= \max\{\operatorname{reg}(J_1), \operatorname{reg}(K_1), \operatorname{reg}(J_1 \cap K_1) - 1\} \\ &= \max\{m, m, m + (m - l) - 1\} = m + (m - l - 1).\end{aligned}$$

This settles the case $k = 2$.

Suppose that $k \geq 3$ and that the statement holds for all $I_{m,l,t}$ with $t < k$. We consider the ideals $L_0 = I_{m,l,k}$ and $L_i = I_{m,l,k-i-1} + (\prod_{j=1}^{m-l} x_{(k-i-1)(m-l)+j})$ for any $1 \leq i \leq k - 2$. We denote $J_i = I_{m,l,k-i}$ for $1 \leq i \leq k - 1$, $K_1 = (u_k)$, $K_i = (\prod_{j=1}^{m-l} x_{(k-i)(m-l)+j})$ for $2 \leq i \leq k - 1$. Similar to the case $k = 2$, we get that, for $1 \leq i \leq k - 2$, $L_i = J_{i+1} + K_{i+1}$ is a Betti splitting. Notice that $J_i \cap K_i = K_i L_i$, for any $1 \leq i \leq k - 2$, $J_{k-1} \cap K_{k-1} = K_{k-1}(\prod_{j=1}^{m-l} x_j)$ and the fact that the variables that appear in K_i and L_i are different and none of the variables that divide K_{k-1} divide any generator of $\prod_{j=1}^{m-l} x_j$, we obtain that, for $1 \leq i \leq k - 2$,

$$\begin{aligned}\operatorname{pd}(J_i \cap K_i) &= \operatorname{pd}(L_i) = \max\{\operatorname{pd}(J_{i+1}), \operatorname{pd}(K_{i+1}), \operatorname{pd}(J_{i+1} \cap K_{i+1}) + 1\}, \\ \operatorname{reg}(L_i) &= \max\{\operatorname{reg}(J_{i+1}), \operatorname{reg}(K_{i+1}), \operatorname{reg}(J_{i+1} \cap K_{i+1}) - 1\}, \\ \operatorname{reg}(J_i \cap K_i) &= \operatorname{reg}(K_i L_i) = \operatorname{reg}(K_i) + \operatorname{reg}(L_i) \geq \operatorname{reg}(K_i) + 1, \\ \operatorname{reg}(J_{k-1} \cap K_{k-1}) &= \operatorname{reg}(K_{k-1}) + \operatorname{reg}((\prod_{j=1}^{m-l} x_j)) = 2(m - l).\end{aligned}\tag{1}$$

Since $J_{k-1} \cap K_{k-1}$ and K_i are principal ideals, $\operatorname{pd}(J_{k-1} \cap K_{k-1}) = \operatorname{pd}(K_i) = 0$ for $1 \leq i \leq k - 1$. By repeated use of the above equalities (1), the induction assumption

$\text{pd}(J_i) = k - i - 1$, $\text{reg}(J_i) = (k - i - 1)(m - l - 1) + m$ and $m - 2l - 1 \geq 0$, we obtain that $\text{pd}(J_1 \cap K_1) = \text{pd}(L_1) = k - 2$ and $\text{reg}(J_1 \cap K_1) = (k - 1)(m - l - 1) + m + 1$. It follows that

$$\begin{aligned} \text{pd}(L_0) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\ &= \max\{k - 2, 0, k - 1\} = k - 1, \\ \text{reg}(L_0) &= \max\{\text{reg}(J_1), \text{reg}(K_1), \text{reg}(J_1 \cap K_1) - 1\} \\ &= \max\{(k - 2)(m - l - 1) + m, m, (k - 1)(m - l - 1) + m + 1 - 1\} \\ &= (k - 1)(m - l - 1) + m, \end{aligned}$$

□

As a consequence of the above theorem, we have:

Corollary 3.6. *Let k, l, m, n and $I_{m,l,k}$ be as in Theorem 3.5, Then*

$$\text{depth}(I_{m,l,k}) = n - k + 1.$$

Proof. By Auslander-Buchsbaum formula, it follows that

$$\text{depth}(I_{m,l,k}) = n - \text{pd}(I_{m,l,k}) = n - k + 1.$$

□

The following theorem generalizes Theorem 4.1 of [12] and Corollary 4.14 of [1].

Theorem 3.7. *Let k, l, m, n be integers such that $n = k(m - l) + l$ where $k \geq 1$, $m \geq 2$ and $\lceil \frac{m}{2} \rceil \leq l < m$. Let $I_{m,l,k} = (u_1, \dots, u_k)$ with $u_i = \prod_{j=1}^m x_{(i-1)(m-l)+j}$ for any $1 \leq i \leq k$. If $m \equiv 0 \pmod{m-l}$ and we can write n as $n = p(2m - l) + d$ where $0 \leq d < 2m - l$, then*

$$\begin{aligned} (1) \quad \text{pd}(I_{m,l,k}) &= \begin{cases} 2p - 1 & \text{if } d \neq m; \\ 2p & \text{if } d = m. \end{cases} \\ (2) \quad \text{reg}(I_{m,l,k}) &= \begin{cases} p(2m - l - 2) + 1 & \text{if } d \neq m; \\ p(2m - l - 2) + m & \text{if } d = m. \end{cases} \end{aligned}$$

Proof. Let $t = \frac{2m-l}{m-l}$, then $t > 2$. In fact, if $t = 2$, then $l = 0$, contradicting the assumption that $l \geq \lceil \frac{m}{2} \rceil$. We prove these assertions by induction on k .

The cases $k = 1, 2$ are from Theorem 3.5. Suppose that $k \geq 3$ and that the statements hold for all $I_{m,l,s}$ with $s < k$. If $3 \leq k \leq t$, then $n = (2m - l) + d$ with $d = (k - 2)(m - l) < m$.

Set $J_1 = I_{m,l,k-1}$ and $K_1 = (u_k)$, we get that $J_1 \cap K_1 = K_1(\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})$. Thus

$\text{pd}(J_1 \cap K_1) = 0$ and $\text{reg}(J_1 \cap K_1) = m + (m - l) = 2m - l$. As the number of the variables that appear in J_1 is $(2m - l) + d - (m - l)$, using the induction hypothesis, $\text{pd}(J_1) = 1$ and $\text{reg}(J_1) = 2m - l - 1$. It follows that $\text{pd}(I_{m,l,k}) = \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} = 1$, and $\text{reg}(I_{m,l,k}) = \max\{\text{reg}(J_1), \text{reg}(K_1), \text{reg}(J_1 \cap K_1) - 1\} = \max\{2m - l - 1, m, 2m - l - 1\} = 2m - l - 1$. This proves the assertion for $3 \leq k \leq t$.

If $k \geq qt + 1$ with $q \geq 1$. Set $J_1 = I_{m,l,k-1}$ and $K_1 = (u_k)$. By similar arguments as in Theorem 3.5, we get that $I_{m,l,k} = J_1 + K_1$ is a Betti splitting and $J_1 \cap K_1 = K_1(I_{m,l,k-t} + (\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j}))$. Notice that the variables that appear in $K_1, I_{m,l,k-t}$

and $(\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})$ are different, it follows that

$$\begin{aligned} \text{pd}(J_1 \cap K_1) &= \text{pd}(I_{m,l,k-t} + (\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})) \\ &= \text{pd}(I_{m,l,k-t}) + \text{pd}(\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j}) + 1 \\ &= \text{pd}(I_{m,l,k-t}) + 1. \end{aligned}$$

where the second equality follows from Lemma 3.4 (1). We distinguish three cases:

(1) If $k-1 = qt$ with $q \geq 1$, then the numbers of the variables that appear in J_1 and $I_{m,l,k-t}$ are $p(2m-l)+l$ and $(p-1)(2m-l)+m$, respectively. By inductive assumption, we get that $\text{pd}(J_1) = 2p-1$, $\text{pd}(I_{m,l,k-t}) = 2(p-1)$, $\text{reg}(J_1) = p(2m-l-2)+1$ and $\text{reg}(I_{m,l,k-t}) = (p-1)(2m-l-2)+m$. Thus

$$\begin{aligned} \text{pd}(J_1 \cap K_1) &= \text{pd}(I_{m,l,k-t}) + 1 = 2p-1, \\ \text{pd}(I_{m,l,k}) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\ &= \max\{2p-1, 0, (2p-1) + 1\} = 2p; \\ \text{reg}(J_1 \cap K_1) &= \text{reg}(K_1) + \text{reg}(I_{m,l,k-t} + (\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})) \\ &= \text{reg}(K_1) + \text{reg}(I_{m,l,k-t}) + \text{reg}(\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j}) - 1 \\ &= m + [(p-1)(2m-l-2) + m] + (m-l) - 1 \\ &= p(2m-l-2) + m + 1, \\ \text{reg}(I_{m,l,k}) &= \max\{\text{reg}(J_1), \text{reg}(K_1), \text{reg}(J_1 \cap K_1) - 1\} \\ &= \max\{p(2m-l-2) + 1, m, p(2m-l-2) + m\} \\ &= p(2m-l-2) + m. \end{aligned}$$

(2) If $k-1 = qt + 1$ with $q \geq 1$, then the numbers of the variables that appear in J_1 and $I_{m,l,k-t}$ are $(p-1)(2m-l)+m$ and $(p-1)(2m-l)$, respectively. Thus by induction, we have that $\text{pd}(J_1) = 2(p-1)$, $\text{pd}(I_{m,l,k-t}) = 2(p-1)-1$, $\text{reg}(J_1) = (p-1)(2m-l-2)+m$ and $\text{reg}(I_{m,l,k-t}) = (p-1)(2m-l-2)+1$. Therefore, similar to the above assertions, we obtain that $\text{pd}(I_{m,l,k}) = 2p-1$ and $\text{reg}(I_{m,l,k}) = p(2m-l-2)+1$.

(3) If $k-1 = qt + c$ with $q \geq 1$ and $2 \leq c < t$, then the numbers of the variables that appear in J_1 and $I_{m,l,k-t}$ are $p(2m-l) + (c-2)(m-l)$ and $(p-1)(2m-l) + (c-1)(m-l)$, respectively. Thus by induction, we have that $\text{pd}(J_1) = 2p-1$, $\text{pd}(I_{m,l,k-t}) = 2(p-1)-1$, $\text{reg}(J_1) = p(2m-l-2)+1$ and $\text{reg}(I_{m,l,k-t}) = (p-1)(2m-l-2)+1$. Similarly, we can conclude that $\text{pd}(I_{m,l,k}) = 2p-1$ and $\text{reg}(I_{m,l,k}) = p(2m-l-2)+1$. We completed the proof. \square

Remark 3.8. *Theorem 4.1 of [12] and Corollary 4.14 of [1] are some corollaries of the above theorem by specializing to the case that $l = m-1$.*

As another corollary, we obtain the following result:

Corollary 3.9. *Let k, l, m, n and $I_{m,l,k}$ be as in Theorem 3.7, Then*

$$\text{depth}(I_{m,l,k}) = n + 2 - \lceil \frac{n+(m-l)}{2m-l} \rceil - \lfloor \frac{n+(m-l)}{2m-l} \rfloor.$$

Proof. Let $k-1 = qt + c$, where $q \geq 0$ and $0 \leq c < t$. From the proof of the theorem, we get that if $c = 0$, then $d = m$, otherwise, $d = (c-1)(m-l)$. Thus, by some straightforward computations, we have that if $c = 0$, then $\lceil \frac{n+(m-l)}{2m-l} \rceil = \lfloor \frac{n+(m-l)}{2m-l} \rfloor = p+1$, otherwise,

$\lceil \frac{n+(m-l)}{2m-l} \rceil = p+1$ and $\lfloor \frac{n+(m-l)}{2m-l} \rfloor = p$. By Auslander-Buchsbaum formula, we obtain that $\text{depth}(I_{m,l,k}) = n - \text{pd}(I_{m,l,k})$, the desired conclusion follows. \square

Theorem 3.10. *Let k, l, m, n be integers such that $n = k(m-l) + l$ where $k \geq 1$, $m \geq 2$ and $\lceil \frac{m}{2} \rceil \leq l < m$. Let $I_{m,l,k} = (u_1, \dots, u_k)$ with $u_i = \prod_{j=1}^m x_{(i-1)(m-l)+j}$ for any $1 \leq i \leq k$. If $m \equiv s \pmod{m-l}$ with $1 \leq s < m-l$ and we can write n as $n = p(2m-l-s) + d$ where $0 \leq d < 2m-l-s$, then*

$$\text{pd}(I_{m,l,k}) = \begin{cases} 2p-1 & d \neq m; \\ 2p & d = m. \end{cases}$$

Proof. Let $t = \frac{2m-l-s}{m-l}$, then $t > 2$ by similar arguments as in Theorem 3.5. We prove these conclusions by induction on k .

The cases $k = 1, 2$ are from Theorem 3.5. Suppose that $k \geq 3$ and that the statements hold for all $I_{m,l,s}$ with $s < k$. If $3 \leq k \leq t$, then $n = (2m-l-s) + d$ with $s + (m-l) \leq d < m$.

Set $J_1 = I_{m,l,k-1}$ and $K_1 = (u_k)$, we get that $J_1 \cap K_1 = K_1(\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})$. Thus $\text{pd}(J_1 \cap K_1) = 0$. As the number of the variables that appear in J_1 is $(2m-l-s) + d - (m-l)$, using the induction hypothesis, $\text{pd}(J_1) = 1$. It follows that $\text{pd}(I_{m,l,k}) = \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} = 1$. This proves the assertion for $3 \leq k \leq t$.

If $k \geq qt + 1$ with $q \geq 1$. We consider the ideals $L_0 = I_{m,l,k}$, $J_1 = I_{m,l,k-1}$, $K_1 = (u_k)$, $L_1 = I_{m,l,k-t} + (\prod_{j=1}^{m-l} x_{(k-2)(m-l)+j})$, $J_{2q} = I_{m,l,k-qt}$, $K_{2q} = (\prod_{j=1}^{m-l} x_{[k-(q-1)t-2](m-l)+j})$, and for $1 \leq i \leq q-1$,

$$\begin{aligned} J_{2i} &= I_{m,l,k-it}(\Gamma), \\ J_{2i+1} &= I_{m,l,k-it-1}(\Gamma), \\ K_{2i} &= (\prod_{j=1}^{m-l} x_{[k-(i-1)t-2](m-l)+j}), \\ K_{2i+1} &= (\prod_{j=1}^{(t-1)(m-l)} x_{(k-it-1)(m-l)+j}), \\ L_{2i} &= I_{m,l,k-it-1}(\Gamma) + (\prod_{j=1}^{(t-1)(m-l)} x_{(k-it-1)(m-l)+j}), \\ L_{2i+1} &= I_{m,l,k-(i+1)t}(\Gamma) + (\prod_{j=1}^{m-l} x_{(k-it-2)(m-l)+j}). \end{aligned}$$

By similar arguments as in Theorem 3.5, we obtain that, for any $1 \leq i \leq 2q$, we get that $L_i = J_{i+1} + K_{i+1}$ is a Betti splitting and $J_i \cap K_i = K_i L_i$. Notice that the variables that appear in K_i and L_i are different, we obtain that, for any $1 \leq i \leq 2q-1$,

$$\text{pd}(J_i \cap K_i) = \text{pd}(L_i) = \max\{\text{pd}(J_{i+1}), \text{pd}(K_{i+1}), \text{pd}(J_{i+1} \cap K_{i+1}) + 1\}.$$

There are three cases to consider:

(1) If $k-1 = qt$ for some $q \geq 1$, then $n = k(m-l) + l = (qt+1)(m-l) + l = qt(m-l) + m = q(2m-l-s) + m$. By comparing this with the equality $n = p(2m-l-s) + d$, we have that $q = p$ and $d = m$. The numbers of the variables that appear in J_1 and J_{2q} are $p(2m-l-s) + l$ and m , respectively. Similarly, for any $1 \leq i \leq q-1$, the numbers of the variables that appear in J_{2i+1} and J_{2i} are $(p-i)(2m-l-s) + l$ and $(p-i)(2m-l-s) + m$, respectively. Hence, by inductive assumption, $\text{pd}(J_1) = 2p-1$, $\text{pd}(J_{2q}) = 0$, $\text{pd}(J_{2i+1}) = 2(p-i)-1$ and $\text{pd}(J_{2i}) = 2(p-i)$ for $1 \leq i \leq q-1$. Note that $J_{2q} \cap K_{2q} = K_{2q}(\prod_{j=1}^{(t-1)(m-l)} x_j)$ and K_i for $1 \leq i \leq 2q$ are principal ideals, we get that $\text{pd}(J_{2q} \cap K_{2q}) = \text{pd}(K_i) = 0$. By repeated

use of the equality $\text{pd}(J_i \cap K_i) = \max\{\text{pd}(J_{i+1}), \text{pd}(K_{i+1}), \text{pd}(J_{i+1} \cap K_{i+1}) + 1\}$ for $i = 2q-1, 2q-2, \dots, 1$, we obtain that $\text{pd}(J_1 \cap K_1) = 2p-1$. Therefore

$$\begin{aligned} \text{pd}(I_{m,l,k}) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\ &= \max\{2p-1, 0, (2p-1) + 1\} = 2p. \end{aligned}$$

This settles the case $k-1 = qt$ for some $q \geq 1$.

(2) If $k-1 = qt+1$ for some $q \geq 1$, then, similar to the case (1), we have that $q = p+1$ and $d = s$. In this case, the numbers of the variables that appear in J_1 and J_{2q} are $(p-1)(2m-l-s) + m$ and $1 \cdot (2m-l-s) + s$, respectively. Similarly, for any $1 \leq i \leq q-1$, the numbers of the variables that appear in J_{2i+1} and J_{2i} are $(p-i-1)(2m-l-s) + m$ and $(p-i)(2m-l-s) + s$, respectively. Hence, by inductive assumption, $\text{pd}(J_1) = 2(p-1)$, $\text{pd}(J_{2q}) = 1$, $\text{pd}(J_{2i+1}) = 2(p-i-1)$ and $\text{pd}(J_{2i}) = 2(p-i)-1$ for $1 \leq i \leq q-1$. Let

$$\begin{aligned} L_{2q} &= I_{m,l,k-qt-1} + \left(\prod_{j=1}^{(t-1)(m-l)} x_{(k-qt-1)(m-l)+j} \right), J_{2q+1} = I_{m,l,k-qt-1} = I_{m,l,1}, K_{2q+1} = \\ &\left(\prod_{j=1}^{(t-1)(m-l)} x_{(m-l)+j} \right), \text{ then } L_{2q} = J_{2q+1} + K_{2q+1} \text{ is a Betti splitting and } J_{2q} \cap K_{2q} = K_{2q} L_{2q}. \end{aligned}$$

Note that $J_{2q+1} \cap K_{2q+1} = K_{2q+1} \left(\prod_{j=1}^{m-l} x_j \right)$ and K_i for $1 \leq i \leq 2q+1$ are principal ideals, we get that $\text{pd}(J_{2q+1} \cap K_{2q+1}) = \text{pd}(K_i) = 0$. By repeated use of the equality $\text{pd}(J_i \cap K_i) = \max\{\text{pd}(J_{i+1}), \text{pd}(K_{i+1}), \text{pd}(J_{i+1} \cap K_{i+1}) + 1\}$ for $i = 2q, 2q-1, \dots, 1$, we obtain that $\text{pd}(J_1 \cap K_1) = 2(p-1)$. Therefore

$$\begin{aligned} \text{pd}(I_{m,l,k}) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\ &= \max\{2(p-1), 0, 2(p-1) + 1\} = 2p-1. \end{aligned}$$

This settles the case $k-1 = qt+1$ for some $q \geq 1$.

(3) If $k-1 = qt+c$ for some $q \geq 1$ and $2 \leq c < t$, then, similar to the case (1), we have that $p = q+1$ and $d = s + (c-1)(m-l)$.

We claim: $d \neq m$. If $d = m$, then $c-1 = \frac{m-s}{m-l} = t-1$. This implies that $c = t$, contradicting the assumption that $c < t$. This implies $s+(m-l) \leq d < m-l+(c-1)(m-l) < t(m-l)$.

In this situation, the numbers of the variables that appear in J_1 and J_{2q} are $p(2m-l-s) + s + (c-2)(m-l)$ and $1 \cdot (2m-l-s) + s + (c-1)(m-l)$, respectively. Similarly, for any $1 \leq i \leq q-1$, the numbers of the variables that appear in J_{2i+1} and J_{2i} are $(p-i)(2m-l-s) + s + (c-2)(m-l)$ and $(p-i)(2m-l-s) + s + (c-1)(m-l)$, respectively. Hence, by inductive assumption, $\text{pd}(J_1) = 2p-1$, $\text{pd}(J_{2q}) = 1$, $\text{pd}(J_{2i+1}) = 2(p-i)-1$ and

$$\text{pd}(J_{2i}) = 2(p-i)-1 \text{ for } 1 \leq i \leq q-1. \text{ Let } L_{2q} = I_{m,l,k-qt-1} + \left(\prod_{j=1}^{(t-1)(m-l)} x_{(k-qt-1)(m-l)+j} \right),$$

$$J_{2q+1} = I_{m,l,k-qt-1} = I_{m,l,c}, K_{2q+1} = \left(\prod_{j=1}^{(t-1)(m-l)} x_{c(m-l)+j} \right), \text{ then } L_{2q} = J_{2q+1} + K_{2q+1} \text{ is a}$$

Betti splitting, $J_{2q} \cap K_{2q} = K_{2q} L_{2q}$ and $J_{2q+1} \cap K_{2q+1} = K_{2q+1} \left(\prod_{j=1}^{m-l} x_j \right)$. Similar to the above

case (2), we get that $\text{pd}(J_{2q+1} \cap K_{2q+1}) = \text{pd}(K_i) = 0$. By repeated use of the equality $\text{pd}(J_i \cap K_i) = \max\{\text{pd}(J_{i+1}), \text{pd}(K_{i+1}), \text{pd}(J_{i+1} \cap K_{i+1}) + 1\}$ for $i = 2q, 2q-1, \dots, 1$, we can conclude that $\text{pd}(J_1 \cap K_1) = 2(p-1)$. Therefore

$$\begin{aligned} \text{pd}(I_{m,l,k}) &= \max\{\text{pd}(J_1), \text{pd}(K_1), \text{pd}(J_1 \cap K_1) + 1\} \\ &= \max\{2p-1, 0, 2(p-1) + 1\} = 2p-1. \end{aligned}$$

The proof is completed. \square

An immediate consequence of the above theorem is the following:

Corollary 3.11. *Let k, l, m, n, s and $I_{m,l,k}$ be as in Theorem 3.10. Then*

$$\text{depth}(I_{m,l,k}) = n + 2 - \lceil \frac{n + m - l - s}{2m - l - s} \rceil - \lfloor \frac{n + m - l - s}{2m - l - s} \rfloor.$$

Proof. Let $k - 1 = qt + c$, where $q \geq 0$ and $0 \leq c < t$. From the proof of the theorem, we get that if $c = 0$, then $d = m$, otherwise, $d = s + (c - 1)(m - l)$. Thus, by some straightforward computations, we have that if $c = 0$, then $\lceil \frac{n + (m - l - s)}{2m - l - s} \rceil = \lfloor \frac{n + (m - l - s)}{2m - l - s} \rfloor = p + 1$, otherwise, $\lceil \frac{n + (m - l - s)}{2m - l - s} \rceil = p + 1$ and $\lfloor \frac{n + (m - l - s)}{2m - l - s} \rfloor = p$. By Auslander-Buchsbaum formula, we obtain that $\text{depth}(I_{m,l,k}) = n - \text{pd}(I_{m,l,k})$, the desired conclusion follows. \square

To conclude, we ask the following open question.

Problem 3.12. *Let k, l, m, n, s and $I_{m,l,k}$ be as in Theorem 3.10. Does there exist some methods to compute the regularity of the ideal $I_{m,l,k}$?*

Acknowledgments

The author would like to thank Sara Saeedi Madani who point out some mistakes in my the first version.

REFERENCES

- [1] A. Alilooee and S. Faridi, Graded Betti numbers of path ideals of cycles and lines, *Comm. Algebra*, 43, (2015), 5413-5433.
- [2] A. Björner and M. L. Wachs, Shellable nonpure complexes and posets, I, *Trans. Amer. Math. Soc.*, 348 (1996), 1299-1327.
- [3] P. Brumatti and A. F. da Silva, On the symmetric and Rees algebras of (n, k) -cyclic ideals, 16th School of Algebra, Part II (Portuguese) (Brasilia, 2000). *Mat. Contemp.* 21 (2001), 27-42.
- [4] A. Conca and E. De Negri, M-Sequences, graph ideals and ladder ideals of linear type, *J. Algebra*, 211 (1999), 599-624.
- [5] S. Eliahou and M. Kervaire, Minimal resolutions of some monomial ideals, *J. Algebra*, 129 (1990), 1-25.
- [6] S. Faridi, Simplicial trees are sequentially Cohen-Macaulay, *J. Pure Appl. Algebra*, 190 (2003), 121-136.
- [7] S. Faridi, The facet ideal of a simplicial complex, *Manuscripta Math.*, 109 (2002), 159-174.
- [8] G. Fatabbi, On the resolution of ideals of fat points, *J. Algebra*, 242 (2001), 92-108.
- [9] C. A. Francisco, H. T. Hà and A. Van Tuyl, Splittings of monomial ideals, *Proc. Amer. Math. Soc.*, 137 (10), (2009), 3271-3282.
- [10] H. T. Hà and A. Van Tuyl, Splittable ideals and the resolutions of monomial ideals, *J. Algebra*, 309 (1), (2007), 405-425.
- [11] H. T. Hà and A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, *J. Algebraic Combin.*, 27 (2), (2008), 215-245.
- [12] Jing He and A. Van Tuyl, Algebraic properties of the path ideal of a tree, *Comm. Algebra*, 38 (5), (2010), 1725-1742.
- [13] L. T. Hoa and N. D. Tam, On some invariants of a mixed product of ideals. *Arch. Math.*, 94 (4), (2010), 327-337.
- [14] G. Restuccia and R. Villarreal, On the normality of monomial ideals of mixed products, *Comm. Algebra*, 29 (2001), 3571-3580.
- [15] A. V. Tuyl and R. H. Villarreal, Shellable graphs and sequentially Cohen-Macaulay bipartite graphs, *Journal of Combinatorial Theory*, 115 (5), (2008), 799-814.
- [16] R. H. Villarreal, *Monomial algebras*, Dekker, New York, NY, 2001.
- [17] R. H. Villarreal, Cohen-Macaulay graphs, *Manuscripta Math.*, 66 (1990), 277-293.
- [18] Guangjun Zhu, Shellability of simplicial complexes and simplicial complexes with the free vertex property, *Turkish Journal of Mathematics*, 40 (1), (2016), 181-190.